MIT-CTP-2490

UR-1447

MRI-Phy-95-26

hep-ph/9603325

CUTTING RULES AT FINITE TEMPERATURE

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(February 1996)

Abstract

We discuss the cutting rules in the real time approach to finite temperature field theory and show the existence of cancellations among classes of cut graphs which allows a physical interpretation of the imaginary part of the relevant amplitude in terms of underlying microscopic processes. Furthermore, with these cancellations, any calculation of the imaginary part of an amplitude becomes much easier and completely parallel to the zero temperature case.

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I. INTRODUCTION

At zero temperature, the evaluation of the imaginary part of a scattering amplitude is immensely simplified by the use of the cutting (Cutkosky) rules [1] which further make the unitarity of the S-matrix manifest. The fact that the imaginary part of a n-loop amplitude can be related to the on-shell amplitudes with lower order loops is best seen, in the modern language, by the use of the so called largest time equation of 't Hooft and Veltman [2]. Generalizations of these rules to the finite temperature case are essential and useful since a large number of transport coefficients are given by the imaginary part of some equilibrium finite temperature correlators. The cutting rules at finite temperature will not only simplify the evaluation of these coefficients but will also help in understanding the structure of field theories at finite temperature as well as identifying the microscopic processes underlying them. In this paper, we provide such a generalization in the real time approach to finite temperature field theories.

Finite temperature (and density) extensions of the cutting rules have been discussed, in the past, both in the imaginary time formalism [3] as well as in the real time formalism [4]. Explicit imaginary time calculations of the self-energy by Weldon, in various theories, led him to identify the physical meaning of the imaginary part in terms of underlying microscopic processes. On the other hand, the attempt by Kobes and Semenoff to generalize the cutting rules in the real time formalism ran into difficulties beyond the one loop. (Their discussion was completely within the framework of Thermofield Dynamics.) The main reason for this discrepancy appeared to be the fact that large classes of diagrams that vanish at zero temperature (the graphs that cannot be written as cut diagrams) do not vanish at finite temperature due to the distinct form of the finite temperature propagators, and this extra class of diagrams do not admit the physical interpretation sugested by Weldon. Such new graphs arise only at two and higher loops. More recently, Jeon has proposed Cutkosky-like rules for the computation of imaginary parts in the imaginary time approach. It also includes graphs that can not be interpreted as cut diagrams. This raises two possible interesting scenarios. Namely, either the physical interpretation of the imaginary parts or the cutting rules proposed need to be analyzed more carefully.

In this paper we discuss the generalization of the cutting rules for the finite temperature case in the real time approach. Our discussion is within the context of the Closed Time Path formalism where the propagators satisfy various identities which makes the analysis of complicated diagrams a lot easier. We show that while graph by graph there arise new contributions (distinct from the zero temperature case), large classes of graphs cancel among themselves leading to the fact that there are no “extra” diagrams to be considered different from the zero temperature case. The cancellations arise because of the properties of the propagators as well as the KMS conditions present in the theory. This shows, in particular, that the interpretation sugested by Weldon, in fact, holds and that the imaginary part of an amplitude at any loop can be written as a sum over cut diagrams much like the zero temperature case. This has the added advantage that the number of diagrams needed to evaluate the imaginary part reduces considerably and it makes clear the physical meaning of this in terms of underlying microscopic processes. The paper is organized as follows. In section II, we introduce the notion of circling and discuss the generalization of the cutting rules. In section III, we prove the cancellation among classes of diagrams leading to the

correct description of the imaginary part in terms of cut diagrams. In section IV, we discuss this explicitly at one and two loops with the example of a ϕ^3 theory alongwith the physical interpretation and present a short conclusion in section V.

II. CUTTING RULES

In the real time formalism, the number of field degrees of freedom double at finite temperature leading to a 2×2 matrix structure for the Greens functions and the propagators of the theory. In the Closed Time Path formalism, for example, we can write the Greens functions of, say, a scalar theory as $(a, b = +, -)$

$$\Delta_{ab} = \begin{pmatrix} \Delta_{++} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{--} \end{pmatrix} \tag{1}$$

where in momentum space, the Greens functions have the form

$$\Delta_{++}(k) = \frac{1}{k^2 - m^2 + i\epsilon} - 2\pi i n(k_0)\delta(k^2 - m^2)\ , \tag{2a}$$
$$\Delta_{--}(k) = -\frac{1}{k^2 - m^2 - i\epsilon} - 2\pi i n(k_0)\delta(k^2 - m^2)\ , \tag{2b}$$
$$\Delta_{+-}(k) = -2\pi i(n(k_0) + \theta(-k_0))\delta(k^2 - m^2)\ , \tag{2c}$$
$$\Delta_{-+}(k) = -2\pi i(n(k_0) + \theta(k_0))\delta(k^2 - m^2)\ , \tag{2d}$$

with $n(k_0)$ representing the bosonic distribution function at $\beta = 1/kT$

$$n(k_0) = \frac{1}{e^{\beta|k_0|} - 1}.$$

It is interesting to note from the explicit forms of the Greens functions above that they satisfy the identity

$$\Delta_{++} + \Delta_{--} = \Delta_{+-} + \Delta_{-+}. \tag{3}$$

We can now arrive at the finite temperature generalization of the cutting rules following closely the argument presented in [2]. The Källen-Lehman spectral representation for the Green's function Δ_{ab}, at finite temperature, has the form [5]

$$\Delta_{ab}(x) = \int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} \left[\frac{\rho_{ab}(s, \vec{k})}{k^2 - s + i\epsilon} + \frac{\tilde{\rho}_{ab}(s, \vec{k})}{k^2 - s - i\epsilon} \right] e^{-ik\cdot x}, \tag{4}$$

where $\rho_{ab}(s, k), \tilde{\rho}_{ab}(s, k)$ are the spectral functions. We can perform the integral over k_0 and it is easy now to see that $\Delta_{ab}(x)$ can be written as

$$\Delta_{ab}(x) = \theta(x^0)\Delta^+_{ab}(x) + \theta(-x^0)\Delta^-_{ab}(x), \tag{5}$$

where the functions $\Delta^\pm_{ab}(x)$ are defined by

$$\Delta^\pm_{ab}(x) = \int_0^\infty ds \int \frac{d^4k}{(2\pi)^4}\, 2\pi i\delta(k^2 - m^2) \left[-\theta(\pm k_0)\rho(s, \vec{k}) + \theta(\mp k_0)\tilde{\rho}(s, \vec{k}) \right] e^{-ik\cdot x}. \tag{6}$$

We note here that the spectral functions can be read out from the structure of the propagators in momentum space to be

$$\rho_{++}(s,k) = \delta(s-m^2)(1+n(k_0)), \quad \tilde{\rho}_{++}(s,k) = -\delta(s-m^2)n(k_0) \tag{7a}$$
$$\rho_{--}(s,k) = \delta(s-m^2)n(k_0), \quad \tilde{\rho}_{--}(s,k) = -\delta(s-m^2)(1+n(k_0)) \tag{7b}$$
$$\rho_{+-}(s,k) = -\tilde{\rho}_{+-}(s,k) = \delta(s-m^2)(\theta(-k_0)+n(k_0)), \tag{7c}$$
$$\rho_{-+}(s,k) = -\tilde{\rho}_{-+}(s,k) = \delta(s-m^2)(\theta(k_0)+n(k_0)). \tag{7d}$$

The fact that $\rho(s,k) = -\tilde{\rho}(s,k)$ for the $(+-)$ and $(-+)$ functions reflects the fact that those functions, being solutions of the homogeneous equations of motion, are regular at x^0. This brings out the distinctive feature at finite temperature, namely, that, unlike the case at zero temperature, the functions $\Delta^{\pm}_{ab}(x)$ contain both positive and negative frequencies. This is what leads to additional contributions at finite temperature to the imaginary part of an amplitude graph by graph.

Next, we define the notion of circling much the same way as at zero temperature. A propagator with one of the ends circled is defined by replacing $\Delta_{ab}(x-y)$ with $\Delta^{+}_{ab}(x-y)$ if the vertex x is circled and with $\Delta^{-}_{ab}(x-y)$ if the vertex y is circled. A propagator with both the ends circled, namely, if both x and y are circled, is defined by replacing Δ_{ab} by $\tilde{\Delta}_{ab}(x-y) = \theta(x^0)\Delta^-(x-y) + \theta(-x^0)\Delta^+(x-y)$. In addition we also assign an extra factor of -1 to each circled vertex. These rules are summarized in figure 1.

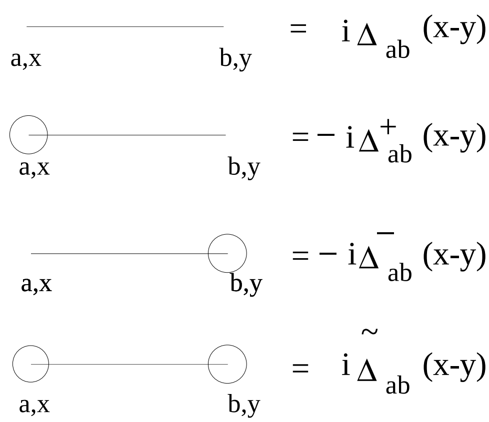

Figure 1

More explicitly, the four components of the propagator of the theory, $\Delta_{++}, \Delta_{--}, \Delta_{+-}$ and Δ_{-+}, have the following circled representations (see equations 6 and 7a).

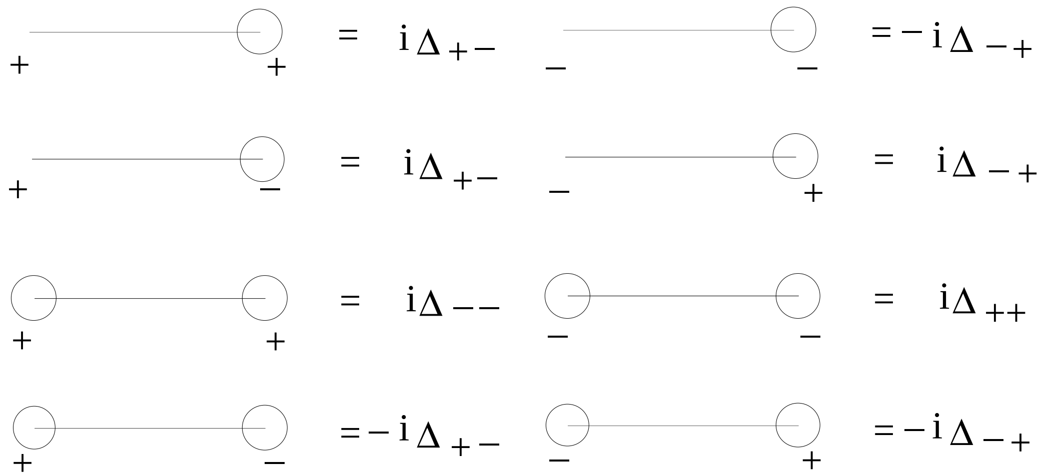

Figure 2

In addition to the identity in eq. (3), the propagators at finite temperature satisfy other relations as well. Thus, for example, the KMS condition would say that

$$\Delta_{ab}(t,\vec{x}) = \Delta_{ab}(t - i\beta, \vec{x}) \tag{8}$$

which, in momentum space, leads to

$$\begin{aligned}\Delta_{+-}(k) &= e^{-\beta k_0}\Delta_{-+}(k)\\ \Delta_{-+}(k) &= e^{\beta k_0}\Delta_{+-}(k).\end{aligned} \tag{9}$$

Furthermore, from the definitions, it is easy to show that

$$(i\Delta_{ab})^*(p) = i\tilde{\Delta}_{ab}(p) \tag{10}$$

We note that these are only some of the relations that will be useful in our analysis. The propagators satisfy various other relations which can be obtained from their definitions.

From these discussions, we arrive at the largest time equation much like at zero temperature. The largest time equation says that a graph with vertices $x_1, ..., x_n$ plus the same graph with the vertex with the largest time circled is zero. (It is easy to see from the definitions that there exists a corresponding smallest time equation as well, but we will not use it in our analysis.)

$$G(x_1, ..., x_k, ..., x_n) + G(x_1, ..., \mathbf{x_k}, ..., x_n) = 0, \quad x_k^0 > x_1^0, ...x_n^0, \tag{11}$$

where a bold face vertex stands for a circled vertex. Equation (11) is a direct consequence of the definitions of figure 1. Notice that (11) is true regardless of whether other vertices besides x_k are circled. From (11) it now follows that the sum of any graph over all possible circlings must vanish. Namely,

$$G(x_1,...,x_k,...,x_n) + G(\mathbf{x_1},...,\mathbf{x_k},...,\mathbf{x_n}) + \sum_{\text{circlings}} G(x_1,...,x_k,...,x_n) = 0, \tag{12}$$

where the sum, in the last term, is over all possible circlings excluding the diagram where all vertices are circled. To show this, we observe that the sum above can be decomposed into a sum over pairs of diagrams with a given set of circlings and with the largest time vertex circled or uncircled. By (11) these pairs vanish and, therefore, the sum adds up pairwise to zero. Using (10) , we now see that we can write the above relation also as

$$\text{Im} \quad iG(p_1,...,p_m) = \frac{1}{2}(iG(p_1,...,p_m) + (iG(p_1,...,p_m))^*) = -\frac{1}{2}\sum_{circlings} iG(p_1,...,p_m). \tag{13}$$

At zero temperature the functions $\Delta^+(k)$ ($\Delta^-(k)$) contain only positive (negative) energies and, using energy conservation at each vertex, it is possible to show that a circled vertex cannot be surrounded by uncircled vertices only (and *vice versa*). In other words, such graphs vanish by energy conservation. (Incidentally, this is how the unphysical degrees of freedom of the finite temperature field theory drop out of physical amplitudes at zero temperature.) As a result, the only nontrivial graphs are those where the circled vertices form a connected region in a diagram that includes one external vertex (that is, a vertex connected to an external line). This leads to the notion of cutting which basically says that in such a case, it is possible to separate the graph into a circled and an uncircled region by drawing a line through the graph (see figure 3).

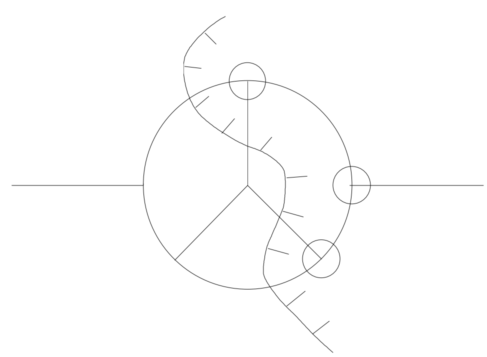

Figure 3

The last term in (13), in this notation, simply becomes a sum over all cut diagrams and, in this case, equation (13) leads to the usual zero temperature Cutkosky rule.

On the other hand, as we mentioned earlier, at finite temperature the functions $\Delta^{\pm}(k)$ contain both positive and negative frequency components and as a result, graphs with isolated circled vertices or isolated uncircled vertices do not vanish anymore. These contribute and should be included in the right hand side of (13). These are the additional graphs that we alluded to earlier and it is not clear any more whether in the presence of these new graphs, the notion of cutting would go through. In the next section, we will argue that even though individually such graphs contribute, there exist classes of such graphs which cancel to leave us with no unwanted diagrams.

III. CANCELLATIONS AMONG GRAPHS

The computation of the imaginary part of any Green's function involves a double sum at finite temperature, namely, we have to sum over all possible circlings as in zero temperature, but in addition we must also sum over the two kinds of intermediate vertices + and − because of the doubled degrees of freedom. Thus, at finite temperature, for any Greens function, we have

$$\text{Im } iG_{a...b} = -\frac{1}{2} \sum_{\text{internal vertices}=\pm} \sum_{\text{circlings}} \text{diagram}, \tag{14}$$

where $a...b = \pm$ are the "thermal indices" of the external legs. As we have mentioned before, unlike the zero tempertaure case, circlings that cannot be represented by a cut diagram as in figure 2 also appear in (14). Examples of such graphs are in figure 6(i,j,k,l,m,n). All such graphs, however, are combinations of the same basic propagators $\Delta_{++}, \Delta_{--}, \Delta_{+-}$ and Δ_{-+} regardless of the combination of + and − vertices, circled or uncircled. We will show now that, after summing over the internal thermal index + and −, there are cancellations among such graphs which makes unnecessary to consider unwanted kinds of circlings. The circlings that we are left with finally, are of the kind of those in figure 2, which can be represented by a cut diagram. This not only simplifies the evaluation of the imaginary parts by reducing the number of graphs to be considered but also allows a physical interpretation in terms of deacy rates and emission/absorption of particles to/from the medium.

To proceed, let us note that any self-energy graph with any combination of circled and uncircled vertices can be drawn in one of the three forms shown in figure 4(a,b,c).

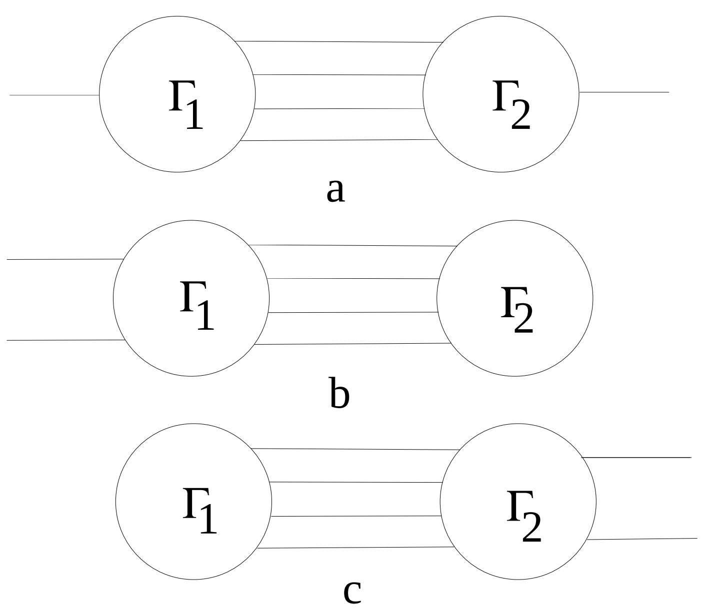

Figure 4

Here, Γ_1 is assumed to contain all the circled vertices of the graph and only the circled vertices while Γ_2 contains all the uncircled vertices of the graph and only the uncircled ones. The vertices in Γ_1 (or Γ_2) need not all be connected to each other. The kind of diagrams that have a non vanishing contribution to the right hand side of (13) at *zero temperature* are the ones that can be drawn as in figure 4(a), with all the vertices in Γ_1 connected with each other, as well with all vertices of Γ_2 connected with each other. We say that graphs of this form can be drawn as a cut graph since a cut between Γ_1 and Γ_2 separates all circled vertices from the uncircled ones, as well as the incoming from the outgoing lines, leaving no island of circled or uncircled vertices isolated from external lines. On the other hand, graphs such as the ones in figure 5(b,c) cannot be represented by a cut diagram since one cannot draw a line separating a circled from uncircled vertices which will also separate the incoming and the outgoing external lines. Graphs like figure 4(a) with Γ_1 (or Γ_2) disconnected can not be represented by a cut diagram either, since a line separating Γ_1 from Γ_2 will also split Γ_1 (Γ_2) in two, leaving an cluster of circled (uncircled) vertices isolated from any external line. We will now show that the diagrams that can not be represented by a cut diagram, in the sense explained above, vanish after a summation of the indices $+$ and $-$ of the internal vertices is performed. This amounts to saying that the kinds of circlings that we need to consider, at finite temperature, are precisely the ones appearing at zero temperature. To show this, we essentially need two main results.

Our first result is that any graph containing a connected cluster of *circled* vertices not attached to any external line must vanish. This result disposes of the graphs of the form of figure 4(c) as well as the ones of the form of figure 5(a) if Γ_1 does not form a connected set of vertices. We can see that by taking this connected cluster of circled vertices isolated from external lines to be Γ_1 in graphs like figure 4(c), or the component of Γ_1 disconnected from the external lines in graphs like figure 4(a).

This result is most easily proven in position space, using a variation of the argument presented in [2]. Let us consider the cluster of circled vertices isolated from the external lines that we assume to exist and let us focus on the circled vertex *inside this cluster* with the *smallest* time of all the vertices in the cluster. Since it is assumed that the external vertices are not circled, this must be an internal vertex and, as such, we should sum over its thermal index $a = \pm$. In general, this vertex will be connected to both $+$ and $-$ vertices, circled and uncircled, as well with itself (forming tadpoles). Let us assume that this smallest time vertex (among the circled vertices) is connected to n uncircled type $+$ vertices at coordinates $y_1, ..., y_n$, m uncircled type $-$ vertices located at $z_1, ..., z_m$, p circled type $+$ vertices located at $r_1, ..., r_p$, q circled type $-$ vertices at $s_1, ..., s_q$ and to itself l times (see figure 5).

Summing over the thermal index of the smallest time circled vertex, and taking into account that

$$\Delta_{++}(x) = \theta(x^0)\Delta_{-+}(x) + \theta(-x^0)\Delta_{+-}(x) \tag{15a}$$
$$\Delta_{--}(x) = \theta(x^0)\Delta_{+-}(x) + \theta(-x^0)\Delta_{-+}(x), \tag{15b}$$

we have

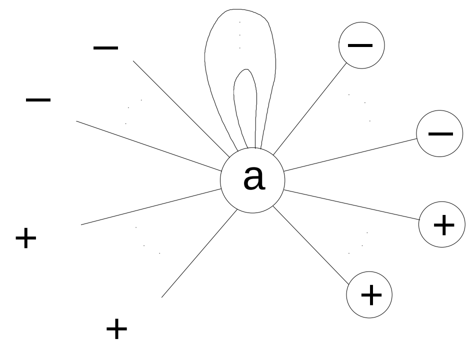

Figure 5

$$
\begin{aligned}
\sum_{a=\pm} \text{diagram} = \quad & R\ \Delta^l_{--}(0)\quad \Delta_{-+}(x-y_1)...\Delta_{-+}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_m)\\
& \qquad\qquad \Delta_{--}(x-r_1)...\Delta_{--}(x-r_p)\Delta_{+-}(x-s_1)...\Delta_{+-}(x-s_q)\\
& - R\ \Delta^l_{++}(0)\Delta_{-+}(x-y_1)...\Delta_{-+}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_m)\\
& \qquad\qquad \Delta_{-+}(x-r_1)...\Delta_{-+}(x-r_p)\Delta_{++}(x-s_1)...\Delta_{++}(x-s_q)\\
= \quad & R\ \Delta^l_{++}(0)\quad \Delta_{-+}(x-y_1)...\Delta_{-+}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_m)\\
& \qquad\qquad [\Delta_{-+}(x-r_1)...\Delta_{-+}(x-r_p)\Delta_{+-}(x-s_1)...\Delta_{+-}(x-s_q)\\
& \qquad\qquad \Delta_{-+}(x-r_1)...\Delta_{-+}(x-r_p)\Delta_{+-}(x-s_1)...\Delta_{+-}(x-s_q)]\\
= \quad & \quad 0, \qquad\qquad (16)
\end{aligned}
$$

where R stands for the rest of the diagram which is not connected to the circled vertex with the smallest time (and contains a multiplicative factor of $i^{m+n+p+q+l}$ coming from the definition of the propagators. In addition, we have used the fact that

$$\Delta_{++}(0) = \Delta_{--}(0), \tag{17}$$

to arrive at the second equality. We should add here that even though the coordinates of the intermediate vertices are being integrated over, this result still holds because there will always be one circled intermediate vertex with the smallest time among the group and the proof would go through.

This proves the first of the two results mentioned above, namely, the sum over the thermal indices of the internal vertices vanishes if there is a connected cluster of circled vertices which does not include the external vertices. As we have pointed out earlier, this includes graphs of the form of figure 4(c) as well as figure 4(a) for disconnected Γ_1.

It is worth stressing here that this result does not apply to graphs containing only one cluster of circled vertices which include at least one external vertex. This is because, in such a case, for some values of the coordinates of the intermediate vertices, the external circled vertex may, in fact, be the one with the smallest time. On the other hand, being an

external vertex, its index is fixed for a particular amplitude and should not be summed and as a result the theorem fails. A further cancellation, however, occurs in the computation of the imaginary part of the retarded self-energy given by $\Sigma_R = \Sigma_{++} - \Sigma_{+-}$. It is clear that evaluation of this would involve, in addition to summing over the thermal indices of the internal vertices, a sum over the thermal index of one of the external vertices (examples are the ones in figure 6(e,f,g,h)). As a result, these graphs should vanish when the external index being summed is a circled vertex by the above theorem.

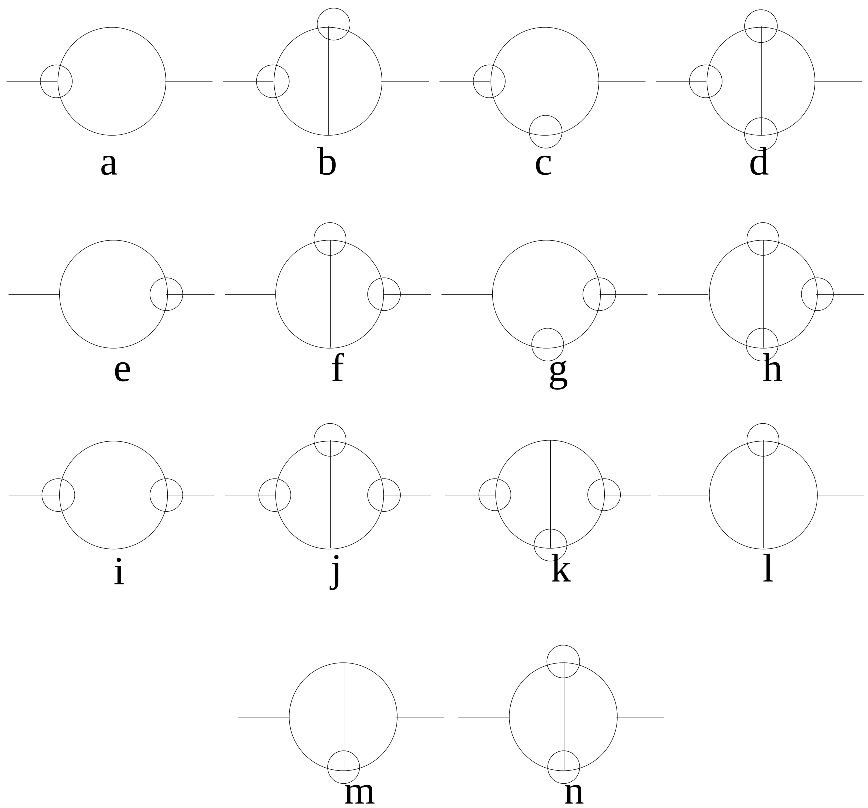

Figure 6

The second result which we need is, in a sense, complementary to the first one. We will show that graphs containing a connected cluster of *uncircled* vertices not connected to any external line vanishes. This result implies that graphs like those of figure 4(b), as well that those graphs like the figure 4(a) with Γ_2 composed of two or more disconnected pieces, must vanish. To see this we just take this connected cluster of uncircled vertices not attached to any external line to be Γ_2, in graphs like figure 4(b), or a component of Γ_2 not attached to the external line, in graphs like figure 4(a).

We can show our second result in the following way. Let $k_1, ..., k_n$ represent the momenta carried by the lines leaving Γ_1 and going into the cluster of uncircled vertices not attached to any external line Λ (Λ is Γ_2 for graphs of the form of figure 4(b), but represents only the component of Γ_2 not attached to any external vertices for graphs like the figure 4(a) with Γ_2

disconnected). The important thing to observe here is that the propagators connecting Γ_1 and Λ do not depend on the thermal indices of the vertices in Γ_1, but only on the vertices on Λ. This is because these are necessarily propagators with only one end circled and as is clear from figure 2 in this case, the propagators have opposite thermal indices labelled completely by the uncircled vertex. Let us consider next a graph of the type of figure 4(b) (or of the type 4(a) with Γ_2 disconnected) for an arbitrary fixed set of thermal indices for the vertices that are not in Λ while we sum over the thermal indices of the Λ. For any such configuration, these graphs factorize as

$$\Sigma(p) = \Gamma(p, k_1, ..., k_n) \sum_{\alpha_1,...,\alpha_n=\pm} \Lambda_{\alpha_1,...,\alpha_n}(k_1, ..., k_n)\Delta_{\alpha_1\ -\alpha_1}(k_1)...\Delta_{\alpha_n\ -\alpha_n}(k_n). \tag{18}$$

where $\Lambda_{\alpha_1,...,\alpha_n}(k_1, ..., k_n)$ and $\Gamma(p, k_1, ..., k_n)$ represent the contributions coming from connecting the vertices in and outside of Λ respectively (as well as factors of i coming from the propagators and vertices). The sum over each index α_i of the vertices in $\Lambda_{\alpha_1,...,\alpha_n}(k_1, ..., k_n)$ can be performed as follows

$$\begin{aligned}
\sum_{\alpha_i=\pm} \Lambda_{\ ...\alpha_i...}\ (k_1, ..., k_n)\ ...\Delta_{\alpha_i\ -\alpha_i}(k_i)\ ... &= \Lambda_{\ ...+...}(k_1, ..., k_n)\ ...\Delta_{-+}(k_i)... \\
&\quad + \Lambda_{\ ...-...}(k_1, ..., k_n)\ ...\Delta_{+-}(k_i)... \\
&= \ ...\Delta_{+-}(k_i)...\ [\Lambda_{\ ...+...}(k_1, ..., k_n)e^{\beta k_i^0} \\
&\qquad + \Lambda_{\ ...-...}(k_1, ..., k_n)],
\end{aligned} \tag{19}$$

where we used the KMS condition of eq.(9)

$$\Delta_{-+}(k) = e^{\beta k_0}\Delta_{+-}(k), \tag{20}$$

to rearrange some of the propagators. The signs associated with the $+$, $-$ vertices as well as the circlings are included in the definition of the Γ's . After the sum over all the indices $\alpha_1, ..., \alpha_n$ we are left with

$$\Sigma(p) = \Gamma(p, k_1, ..., k_n)\Delta_{+-}(k_1)...\Delta_{+-}(k_n) \sum_{\alpha_1,...,\alpha_n=\pm} \Lambda_{\alpha_1,...,\alpha_n}(k_1, ..., k_n)e^{\beta\sum_{j=1}^n \frac{\alpha_j+1}{2}k_j^0}. \tag{21}$$

Let us note here that $\frac{\alpha_j+1}{2} = 1$ or 0 depending on whether $\alpha_j = 1$ or -1. The terms in the sum above combine pairwise to produce

$$\begin{aligned}
\Lambda_{\alpha_1,...,\alpha_n}(k_1, ..., k_n)e^{\beta\sum_{j=1}^n \frac{\alpha_j+1}{2}k_j^0} &+ \Lambda_{-\alpha_1,...,-\alpha_n}(k_1, ..., k_n)e^{\beta\sum_{j=1}^n \frac{-\alpha_j+1}{2}k_j^0} \\
&= \Lambda^*_{\alpha_1,...,\alpha_n}(k_1, ..., k_n) + \Lambda^*_{-\alpha_1,...,-\alpha_n}(k_1, ..., k_n).
\end{aligned} \tag{22}$$

This is easily seen as follows. Let us note that $\Lambda(k_1, ..., k_n)\alpha_1, ..., \alpha_n$ represents a graph containing only uncircled vertices of type $+$ and $-$. Therefore, let us consider a generic (amputated) diagram with n $(+)$ external lines, m $(-)$ external lines, and draw it by grouping the positive (negative) internal vertices into "blobs" Λ_+^n (Λ_-^m), as in figure 7.

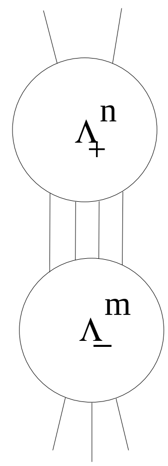

Figure 7

Taking the complex conjugate of this diagram (in momentum space) and using the fact that $(i\Delta_{++})^* = i\Delta_{--})$ and $(i\Delta_{\pm\mp})^* = i\Delta_{\pm\mp})$ as well as the relation 9, we find that (see figure 8) for any number of $+$ and $-$ indices

$$\Lambda^*_{+\ldots+\ -\ldots-} = \Lambda_{-\ldots-\ +\ldots+} e^{-\beta p_0}, \tag{23}$$

where p is the total momentum flowing through $\Gamma_{+\ldots+\ -\ldots-}$, namely, it is the total momentum entering into the diagram of figure 7 through the $+$ vertices. Using this result and combining $\Lambda\ _{\alpha_1,\ldots,\alpha_n}$ and $\Lambda\ _{-\alpha_1,\ldots,-\alpha_n}$ we now arrive at (22).

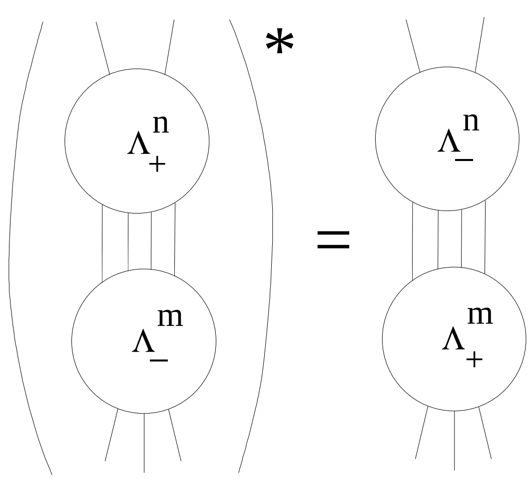

Figure 8

The equation (23) can be regarded as a generalization of the KMS condition for higher point functions. Returning to equation (21) we now have

$$\Sigma(p) = \Gamma(p, k_1, ..., k_n)\Delta_{+-}(k_1)...\Delta_{+-}(k_n) \sum_{\alpha_1,...,\alpha_n=\pm} \Lambda^*{}_{\alpha_1,...,\alpha_n}(k_1, ..., k_n). \tag{24}$$

We can now show easily that the sum above vanishes using the largest/smallest time argument. Let us consider a graph contributing to $\Lambda_{\alpha_1,...,\alpha_n}$ in position space and choose the vertex with the largest time labelling its coordinate and thermal index respectively as x and a. This vertex is connected generically to n type + vertices located at $y_1, ..., y_n$ and m type $-$ vertices located at $z_1, ..., z_n$ (there are no circled vertices in *Lambda*). The sum over $a = \pm$ is easily seen to give

$$\begin{aligned}
\text{diagram} &= R[\Delta_{++}(x-y_1)...\Delta_{++}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_n) \\
&\quad -\Delta_{-+}(x-y_1)...\Delta_{++}(x-y_n)\Delta_{--}(x-z_1)...\Delta_{--}(x-z_n)] \\
&= R[\Delta_{-+}(x-y_1)...\Delta_{-+}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_n) \\
&\quad -\Delta_{-+}(x-y_1)...\Delta_{-+}(x-y_n)\Delta_{+-}(x-z_1)...\Delta_{+-}(x-z_n)] \\
&= 0,
\end{aligned} \tag{25}$$

where R is the remaining of the diagram, independent of the vertex on x.

This completes the proof of the result alluded to earlier, namely, graphs containing a cluster of uncircled vertices not attached to any external line vanishes.

With these two results, then, we see that the cutting rules for the imaginary part of any diagram has the same form as at zero temperature. The unwanted "extra" graphs present at finite temperature cancel among themselves when we sum over the thermal indices of the internal vertices. Thus, for example, it is clear now that of all the graphs in figure 6, the only ones that will give a nontrivial contribution to the evaluation of the imaginary part of the the retarde self-energy are 4(a,b,c,d) ‡.

IV. PHYSICAL INTERPRETATION

Let us illustrate the cutting rule for the imaginary part of a diagram, discussed in the last section, with a very simple example. We will consider the imaginary part of the one loop retarded self-energy for a scalar theory with a ϕ^3 interaction. As we have mentioned earlier, the retarded and the causal self-energies are related by a multiplicative factor and yet we choose to examine the retarded function only because this is the one that has a direct connection with the calculations in the imaginary time formalism. Furthermore, it is the retarded function at finite temperature where the cutting of a diagram leads in a simple way to the unitarity relation. The imaginary part of the retarded self-energy for this theory is given, at one loop, by the graphs in figure 9 (the graphs with the vertex on the right circled cancel) and an easy calculation gives

‡The imaginary part of the causal self energy is just a multiple of the imaginary part of the retarded self energy [5]

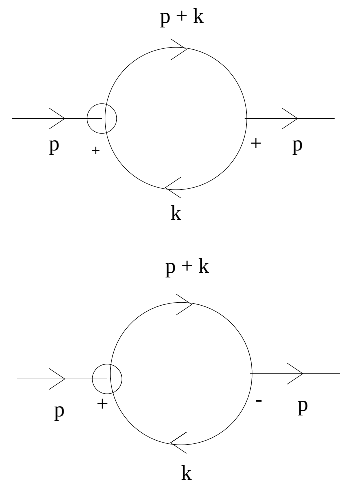

Figure 9

$$
\begin{aligned}
i\Sigma(p) &= -g^2 \int \frac{d^4k}{(2\pi)^4} \left[i\Delta_{+-}(k)\ i\Delta_{-+}(p+k) - i\Delta_{-+}(k)\ i\Delta_{+-}(p+k) \right] \\
&= g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_1 2\omega_2} [\delta(p_0 + \omega_1 - \omega_2)(n_1(n_2+1) - n_2(n_1+1)) \\
&\qquad + \delta(p_0 - \omega_1 - \omega_2)((n_1+1)(n_2+1) - n_2 n_1 + 1) \\
&\qquad + \delta(p_0 + \omega_1 + \omega_2)(n_1 n_2 - (n_2+1)(n_1+1)) \\
&\qquad + \delta(p_0 - \omega_1 + \omega_2)(n_2(n_1+1) - n_1(n_2+1))], \qquad (26)
\end{aligned}
$$

with $\omega_1(\omega_2)$ representing ω_k (ω_{p+k}) and $n_1(n_2)$ representing $n(\omega_k)(n(\omega_{p+k}))$ respectively. This one loop result has been obtained in several ways for various intermediate states (as we have checked also) and the physical interpretation of this result was already pointed out in [6]. As in the zero temperature case, the imaginary part of finite temperature self energies are related to decay probabilities of the incoming particle, but with two differences. First the probabilities of (boson) emission are enhanced by a factor $(n+1)$ dependent on the occupation number of that particle in the medium (stimulated emission) [§]. Second, new processes which do not have an analog in the vacum arise at finite temperature. These are processes involving the absorption of one or more real particles from the medium. The absorption probability, of course, is proportional to n which represents the density of such particles present in the medium.

The possibility of extending this interpretation for higher loop graphs clearly depends crucially on whether we can write the imaginary part of the self-energy as a sum of cut

[§]Pauli blocking would appear if the intermediate states included fermions

graphs. Only those are of the form of a decay amplitude times its complex conjugate **, weighted by statistical factors. That is exactly what we have proved in the previous section. A simple example of how this works in a two loop graph is given pictorially in figure 10.

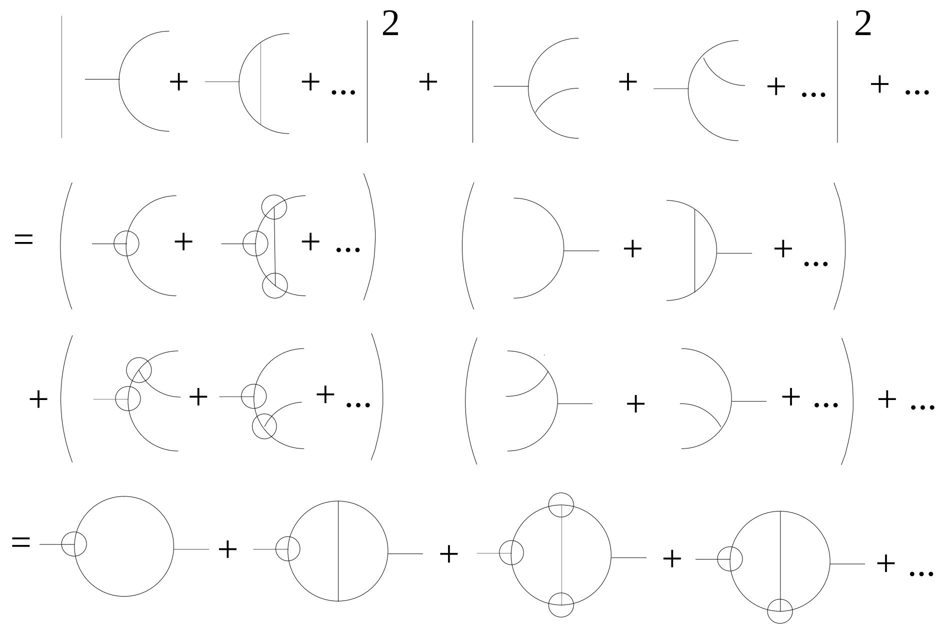

Figure 10

The sum of the probabilities for the decay of the incomimg particle in 2, 3 or more particles (plus the processes involving absorption of the particles from the medium) equals the sum of the circlings in figure 6 that can be drawn as a cut diagram. The remaining graphs in figure 6 would spoil the physical interpretation but they all vanish by our general arguments. This can also be checked explicitly. The graph in figure 6(l), for example, with the right, left and bottom vertices of the type + but the index of the top vertex summed over is given by

$$\begin{aligned}\text{graph 4(l)} &= \Delta_{++}(x-z')\Delta_{++}(y-z')[\Delta_{+-}(x-z)\Delta_{+-}(z'-z)\Delta_{++}(y-z) \\ &\qquad - \Delta_{+-}(x-z)\Delta_{+-}(z'-z)\Delta_{++}(y-z)] \\ &= 0,\end{aligned} \tag{27}$$

where x, y, z, z' are, respectivelly, the position of the left, right, top and bottom vertices. This is a special case of our first main result. As an example of a diagram that vanishes by

**Remember that the complex conjugate of a graph with no circled vertices is the same graph with all vertices circled.

our second main result, let us take the graph in figure 4(j). Fixing, for instance, the circled vertices to be of type $+$, and denoting the momentum flowing from the bottom vertex to the one on the left by k_1, the momentum flowing from the bottom vertex to the one on the right by k_2, and the incoming momentum by p, the sum over the bottom vertex gives

$$\begin{aligned}
\text{graph 4(j)} &= \Delta_{--}(p+k_1)\Delta_{--}(p-k_2)[\Delta_{+-}(k_1)\Delta_{+-}(k_2)_{+-}\Delta_{+-}(-k_1-k_2) \\
&\qquad\qquad\qquad\qquad - \Delta_{-+}(k_1)\Delta_{-+}(k_2)_{-+}\Delta_{-+}(-k_1-k_2)] \\
&= \Delta_{--}(p+k_1)\Delta_{--}(p-k_2)\Delta_{+-}(k_1)\Delta_{+-}(k_2)_{+-}\Delta_{+-}(-k_1-k_2) \\
&\qquad\qquad \times\, (e^{k_1^0+k_2^0-k_1^0-k_2^0}-1) \\
&= 0.
\end{aligned} \tag{28}$$

V. CONCLUSION

We have considered the generalization of diagramatic cutting rules to the finite temperature case using the real time formalism. Graphs that can not be represented by cut diagrams are shown to cancel, and those that are left have a nice interpretation in terms of decay, absorption and emission probabilities. We have concentrated on self-energy graphs but the analysis remain virtually unchanged for graphs with more exeternal lines.

VI. ACKNOWLEDGEMENTS

One of us (A.D.) would like to thank Prof. H. Mani and the members of the Physics group at the Mehta Research Institute, for their hospitality, where this work started. This work was supported in part by U.S. Department of Energy grant Nos. DE-FG-02-91ER40685 and DF-FC02-94ER40818.

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